# Note on a two-species competition-diffusion model with two free boundaries[1]

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**Abstract.** In [3, 9], the authors studied a two-species competition-diffusion model with two free boundaries. The existence, uniqueness and long time behavior of global solution were established. In this note we still discuss the long time behavior of global solution and provide some new results and simpler proofs.

**Keywords:** Competition-diffusion model; Free boundary problem; Long time behavior.

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## 1 Introduction

Recently, Guo & Wu [3] studied the existence and uniqueness of global solution $(u, v, s_1, s_2)$ to the following free boundary problem

$$\begin{cases} u_t - d_1 u_{xx} = r_1 u(1-u-kv), & t>0, \quad 0<x<s_1(t),\\ v_t - d_2 v_{xx} = r_2 v(1-v-hu), & t>0, \quad 0<x<s_2(t),\\ u_x(t,0)=v_x(t,0)=0, & t\ge 0,\\ s_1'(t) = -\mu_1 u_x(t,s_1(t)), \quad s_2'(t) = -\mu_2 v_x(t,s_2(t)), & t\ge 0,\\ u=0 \text{ for } x\ge s_1(t), \quad v=0 \text{ for } x\ge s_2(t), & t\ge 0,\\ u(0,x)=u_0(x), \quad v(0,x)=v_0(x), & x\in[0,\infty),\\ s_1(0)=s_1^0>0, \quad s_2(0)=s_2^0>0, & \end{cases}$$

where the parameters are positive constants, and $u_0(x), v_0(x)$ satisfy

$$\begin{aligned} &u_0\in C^2([0,s_1^0]),\ u_0'(0)=0,\ u_0(x)>0 \ \text{ in } \ [0,s_1^0),\ u_0(x)=0 \ \text{ in } \ [s_1^0,\infty),\\ &v_0\in C^2([0,s_2^0]),\ v_0'(0)=0,\ v_0(x)>0 \ \text{ in } \ [0,s_2^0),\ v_0(x)=0 \ \text{ in } \ [s_2^0,\infty). \end{aligned}$$

Furthermore, Guo & Wu [3] and Wu [9] investigated the long time behavior of $(u, v, s_1, s_2)$ for the cases $0<k<1<h$ and $0<k,\ h<1$, respectively.

By use of the arguments of [6, Theorem 2.1] we can prove that $s_1'(t), s_2'(t)>0$, and

$$(u,v,s_1,s_2)\in C^{1+\frac{\alpha}{2},2+\alpha}(\mathcal{D}_\infty^{s_1})\times C^{1+\frac{\alpha}{2},2+\alpha}(\mathcal{D}_\infty^{s_2})\times[C^{1+\frac{1+\alpha}{2}}([0,\infty))]^2,$$

where $\mathcal{D}_\infty^{s_i}=\{t\ge 0,\ 0\le x\le s_i(t)\}$. Moreover, there exists a positive constant $C$ such that

$$\begin{cases} \|u(t,\cdot)\|_{C^1([0,\,s_1(t)])}, \quad \|v(t,\cdot)\|_{C^1([0,s_2(t)])}\le C, & \forall\ t\ge 1,\\ \|s_1',\ s_2'\|_{C^{\alpha/2}([n+1,n+3])}\le C, & \forall\ n\ge 0. \end{cases} \tag{1}$$

We still study the long time behavior of $(u, v, s_1, s_2)$ and provide some new results and simpler proofs. This short paper can be considered as the supplements of papers [3, 9].

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## 2 Preliminaries

**Proposition 1** ([7, Proposition 2.1]) *Let $d, r, a$ be fixed positive constants. For any given $\varepsilon, L > 0$, there exists $l_\varepsilon > \max\{L, \frac{\pi}{2}\sqrt{d/(ra)}\}$ such that, when a non-negative $C^{1,2}$ function $z$ satisfies*

$$\begin{cases} z_t - dz_{xx} \geq rz(a-z), & t > 0, \quad 0 < x < l_\varepsilon, \\ z_x(t,0) = 0, \quad z(t, l_\varepsilon) \geq 0, & t \geq 0 \end{cases}$$

*and $z(0,x) > 0$ in $(0, l_\varepsilon)$, then $\liminf_{t\to\infty} z(t,x) \geq a - \varepsilon$ uniformly on $[0, L]$.*

**Proposition 2** ([1, 2]) *For any given $d, a, b, \mu > 0$, the problem*

$$\begin{cases} dq'' - cq' + q(a - bq) = 0, \quad 0 < y < \infty, \\ q(0) = 0, \quad q'(0) = c/\mu, \quad q(\infty) = a/b, \\ c \in (0, 2\sqrt{ad}); \quad q'(y) > 0, \quad 0 < y < \infty \end{cases} \tag{2}$$

*has a unique solution $(q, c)$. Denote $\gamma = (\mu, a, b, d)$ and $c = c(\gamma)$. Then $c(\gamma)$ is strictly increasing in $\mu$ and $a$, respectively, and is strictly decreasing in $b$. Moreover,*

$$\lim_{\frac{a\mu}{bd}\to\infty} \frac{c(\gamma)}{\sqrt{ad}} = 2, \qquad \lim_{\frac{a\mu}{bd}\to 0} \frac{c(\gamma)}{\sqrt{ad}} \frac{bd}{a\mu} = \frac{1}{\sqrt{3}}. \tag{3}$$

To simplify the notations, we define

$$c_1(\mu, a) = c(\mu, a, r_1, d_1), \quad c_2(\mu, a) = c(\mu, a, r_2, d_2).$$

If $0 < k < 1$, in view of (3), it is easy to see that

$$\lim_{\mu_1\to\infty} c_1(\mu_1, r_1(1-k)) = 2\sqrt{d_1 r_1(1-k)}, \quad \lim_{\mu_2\to 0} c_2(\mu_2, r_2) = 0.$$

By the monotonicity of $c(\gamma)$ in $\mu$, there exist $\mu_1^*, \mu_2^* > 0$ such that $c_1(\mu_1, r_1(1-k)) > c_2(\mu_2, r_2)$ for all $\mu_1 \geq \mu_1^*$ and $0 < \mu_2 \leq \mu_2^*$. Therefore, $[\mu_1^*, \infty) \times (0, \mu_2^*] \subset \mathcal{A}$, where

$$\mathcal{A} = \{(\mu_1, \mu_2) : \mu_1, \mu_2 > 0, \ c_1(\mu_1, r_1(1-k)) > c_2(\mu_2, r_2)\}. \tag{4}$$

Same as [3, 9], we define $s_i^\infty = \lim_{t\to\infty} s_i(t)$, $i = 1, 2$, and

$$s_1^* = \frac{\pi}{2}\sqrt{\frac{d_1}{r_1}}, \quad s_2^* = \frac{\pi}{2}\sqrt{\frac{d_2}{r_2}}, \quad \tilde{s}_1 = \frac{\pi}{2}\sqrt{\frac{d_1}{r_1(1-k)}} \ \text{ if } k < 1, \quad \tilde{s}_2 = \frac{\pi}{2}\sqrt{\frac{d_2}{r_2(1-h)}} \ \text{ if } h < 1.$$

In order to convenient writing, for any given constant $\tau \geq 0$ and function $f(t)$, we set

$$D_\tau^f = \{(t, x) : t \geq \tau, \ 0 \leq x \leq f(t)\}.$$

## 3 Main results and their proofs

Using the estimates (1) and [5, Lemma 3.1], we have

**Theorem 1** *If $s_1^\infty < \infty$ ($s_2^\infty < \infty$). Then*

$$\lim_{t\to\infty}\max_{[0,s_1(t)]} u(t,\cdot) = 0 \quad \Big(\lim_{t\to\infty}\max_{[0,s_2(t)]} v(t,\cdot) = 0\Big).$$

**Theorem 2** *If $s_2^\infty < \infty$, $s_1^\infty = \infty$ ($s_1^\infty < \infty$, $s_2^\infty = \infty$), then $\lim_{t\to\infty} u(t,x) = 1$ ($\lim_{t\to\infty} v(t,x) = 1$) uniformly in any compact subset of $[0,\infty)$.*

**Proof**. Let $z(t)$ be the unique solution of

$$z' = r_1 z(1-z), \quad t > 0; \quad z(0) = \|u_0\|_{L^\infty}.$$

Then $z(t) \to 1$ as $t \to \infty$. The comparison principle leads to

$$\limsup_{t\to\infty} u(t,x) \le 1 \ \text{ uniformly in } [0,\infty). \tag{5}$$

For any given $0 < \varepsilon, \delta \ll 1$ and $L > 0$, let $l_\varepsilon$ be given by Proposition 1 with $d = d_1$, $r = r_1$ and $a = 1 - k\delta$. Since $s_1^\infty = \infty$, $\lim_{t\to\infty}\max_{[0,s_2(t)]} v(t,\cdot) = 0$ (Theorem 1) and $v = 0$ for $x > s_2(t)$, there exists $T \gg 1$ such that $s_1(t) > l_\varepsilon$ and $v(t,x) < \delta$ in $[T,\infty)\times[0,\infty)$. Thus, $u$ satisfies

$$\begin{cases} u_t - d_1 u_{xx} \ge r_1 u(1 - k\delta - u), \ t \ge T, \quad 0 < x < l_\varepsilon, \\ u_x(t,0) = 0, \quad u(t,l_\varepsilon) \ge 0, \qquad t \ge T. \end{cases}$$

In view of Proposition 1, we have $\liminf_{t\to\infty} u(t,x) \ge 1 - k\delta - \varepsilon$ uniformly on $[0,L]$. The arbitrariness of $L, \varepsilon$ and $\delta$ imply that $\liminf_{t\to\infty} u(t,x) \ge 1$ uniformly in any compact subset of $[0,\infty)$. Remember (5), the desired result is obtained. $\square$

Utilizing the iteration methods used in the proof of [7, Theorem 2.4] we can prove the following theorem and the details will be omitted.

**Theorem 3** *Assume $s_1^\infty = s_2^\infty = \infty$. For any given $L > 0$, the following hold:*
(i) *if $0 < h, k < 1$, then $\lim_{t\to\infty} u(t,x) = \frac{1-k}{1-hk}$, $\lim_{t\to\infty} v(t,x) = \frac{1-h}{1-hk}$ uniformly in $[0,L]$;*
(ii) *if $0 < h < 1 \le k$, then $\lim_{t\to\infty} u(t,x) = 0$, $\lim_{t\to\infty} v(t,x) = 1$ uniformly in $[0,L]$;*
(iii) *if $0 < k < 1 \le h$, then $\lim_{t\to\infty} u(t,x) = 1$, $\lim_{t\to\infty} v(t,x) = 0$ uniformly in $[0,L]$.*

**Theorem 4** (i) *If $s_1^* < s_1^\infty < \infty$, then $s_2^\infty = \infty$. If $s_2^* < s_2^\infty < \infty$, then $s_1^\infty = \infty$. As the consequence, $s_1^\infty < \infty$ and $s_2^\infty > s_2^*$ imply $s_2^\infty = \infty$, $s_2^\infty < \infty$ and $s_1^\infty > s_1^*$ imply $s_1^\infty = \infty$;*
(ii) *If $k < 1$ and $s_1^\infty > \tilde{s}_1$, then $s_1^\infty = \infty$. If $h < 1$ and $s_2^\infty > \tilde{s}_2$, then $s_2^\infty = \infty$.*

**Proof**. (i) We only prove the first conclusion. Because of $s_1^\infty > s_1^*$, there exist $0 < \varepsilon \ll 1$ and $T \gg 1$ such that $1 - k\varepsilon > 0$ and $s_1(t) > \frac{\pi}{2}(\frac{d_1}{r_1(1-k\varepsilon)})^{1/2}$ for all $t \ge T$. Assume on the contrary that $s_2^\infty < \infty$. Then $\lim_{t\to\infty}\max_{[0,s_2(t)]} v(t,\cdot) = 0$ by Theorem 1. There exists $T_1 > T$ such that $v(t,x) < \varepsilon$ in $D_{T_1}^{s_2}$. Therefore, $(u, s_1)$ satisfies

$$\begin{cases} u_t - d_1 u_{xx} \ge r_1 u(1 - u - k\varepsilon), \ t > T_1, \quad 0 < x < s_1(t), \\ u_x(t,0) = u(t,s_1(t)) = 0, \qquad t \ge T_1, \\ s_1'(t) = -\mu_1 u_x(t,s_1(t)), \qquad t \ge T_1. \end{cases}$$

Since $s_1(T_1) > \frac{\pi}{2}(\frac{d_1}{r_1(1-k\varepsilon)})^{1/2}$, we have $s_1^\infty = \infty$ ([2, Theorem 3.4]). A contradiction.

(ii) The proof is similar to that of (i) since the estimate (5) holds true for $v$. Please refer to the proof of the following Theorem 5 for details. $\square$

**Corollary 1** (i) *If one of $s_1^0 \ge s_1^*$ and $s_2^0 \ge s_2^*$ holds, then either $s_1^\infty = \infty$ or $s_2^\infty = \infty$;*

(ii) *If $k, h < 1$ and $s_1^0 \ge \tilde{s}_1$, $s_2^0 \ge \tilde{s}_2$, then $s_1^\infty = s_2^\infty = \infty$.*

Theorem 4 and Corollary 1 are the improvements of [3, Theorem 2] and [9, Theorem 1, Propositions 4 and 5], and our proofs are very simple.

To facilitate writing, for $\tau \ge 0$, we introduce the following free boundary problem

$$\begin{cases} w_t - dw_{xx} = rw(a-w), & t > \tau, \quad 0 < x < g(t), \\ w_x(t,0) = 0, \quad w(t,g(t)) = 0, & t \ge \tau, \\ g'(t) = -\mu w_x(t,g(t)), & t \ge \tau, \\ g(\tau) = g_0, \quad w(\tau,x) = w_0(x), & 0 \le x \le g_0, \end{cases} \tag{6}$$

and set $\Lambda = (\tau, d, r, a, \mu, g_0)$.

**Theorem 5** *Let $0 < k < 1 < h$ and $(\mu_1, \mu_2) \in \mathcal{A}$, where $\mathcal{A}$ is given by (4). If $s_1^\infty > \tilde{s}_1$, then $s_1^\infty = \infty$ and $s_2^\infty < \infty$.*

**Proof**. Note that $k < 1$, $s_1^\infty > \tilde{s}_1$ and $(\mu_1, \mu_2) \in \mathcal{A}$, there exist $0 < \varepsilon_0 \ll 1$ and $t_0 \gg 1$ such that $s_1(t) > \frac{\pi}{2}(\frac{d_1}{r_1 a_\varepsilon})^{1/2}$ and

$$k(1+\varepsilon) < 1, \quad c_1(\mu_1, r_1 a_\varepsilon) > c_2(\mu_2, r_2)$$

for all $t \ge t_0$ and $0 < \varepsilon \le \varepsilon_0$, where $a_\varepsilon = 1 - k(1+\varepsilon)$. Since the estimate (5) holds true for $v$, for each fixed $0 < \varepsilon \le \varepsilon_0$, there exists $t_1 > t_0$ such that $v(t,x) < 1 + \varepsilon$ in $[t_1, \infty) \times [0, \infty)$.

Let $(w_1, g_1)$ be the unique solution of (6) with $\Lambda = (t_1, d_1, r_1, a_\varepsilon, \mu_1, s_1(t_1))$ and $w_0(x) = u(t_1, x)$. Then $s_1(t) \ge g_1(t)$, $u(t,x) \ge w_1(t,x)$ in $D_{t_1}^{g_1}$, and $g_1(\infty) = \infty$ since $g_1(t_1) > \frac{\pi}{2}(\frac{d_1}{r_1 a_\varepsilon})^{1/2}$. Consequently, $s_1^\infty = \infty$. Take advantage of [10, Theorem 3.1], it is deduced that, as $t \to \infty$,

$$g_1(t) - \tilde{c}t \to \tilde{\rho} \in \mathbb{R}, \quad \|w_1(t,x) - \tilde{q}(\tilde{c}t + \tilde{\rho} - x)\|_{L^\infty([0, g_1(t)])} \to 0, \tag{7}$$

where $(\tilde{q}, \tilde{c})$ is the unique solution of (2) with $\gamma = (\mu_1, r_1 a_\varepsilon, r_1, d_1)$, i.e., $\tilde{c} = c_1(\mu_1, r_1 a_\varepsilon)$.

Assume on the contrary that $s_2^\infty = \infty$. We first prove

$$\lim_{t\to\infty} \max_{[0,\infty)} v(t, \cdot) = 0. \tag{8}$$

Let $(w, g)$ be the unique solution of (6) with $\Lambda = (0, d_2, r_2, 1, \mu_2, s_2^0)$ and $w_0(x) = v_0(x)$. Then $w(t,x) \ge v(t,x)$, $g(t) \ge s_2(t)$ in $D_0^{s_2}$. Hence, $g(\infty) = \infty$. It follows from [10, Theorem 3.1] that

$$\lim_{t\to\infty} (g(t) - ct) = \rho \in \mathbb{R}, \quad \text{with } c = c_2(\mu_2, r_2). \tag{9}$$

Thanks to $\tilde{c} > c$, $s_2(t) \le g(t)$ and (9), it deduces that, as $t \to \infty$, $g_1(t) - g(t) \to \infty$ and

$$\min_{x \in [0, s_2(t)]} (\tilde{c}t + \tilde{\rho} - x) \ge \tilde{c}t + \tilde{\rho} - g(t) = (\tilde{c} - c)t + \tilde{\rho} - \rho + o(1) \to \infty.$$

Based on $\tilde{q}(y) \nearrow a_\varepsilon$ as $y \to \infty$, we have $\lim_{t\to\infty} \min_{x\in[0,s_2(t)]} \tilde{q}(\tilde{c}t + \tilde{\rho} - x) = a_\varepsilon$. It then follows, upon using (7), that $\lim_{t\to\infty} \min_{[0,s_2(t)]} w_1(t, \cdot) = a_\varepsilon$. This implies $\liminf_{t\to\infty} \min_{[0,s_2(t)]} u(t,\cdot) \ge a_\varepsilon$ since $g_1(t) > g(t) \ge s_2(t)$ and $u(t,x) \ge w_1(t,x)$ in $D_T^{g_1}$ when $T \gg 1$. The arbitrariness of $\varepsilon$ gives

$$\liminf_{t\to\infty} \min_{[0,s_2(t)]} u(t, \cdot) \ge 1 - k. \tag{10}$$

Take $t_2 > t_1$ such that $u(t,x) \geq 1-k-\varepsilon$ in $D^{s_2}_{t_2}$. Let $\tilde{a}_\varepsilon = 1-h(1-k-\varepsilon)$ and $(w_2, g_2)$ be the unique solution of (6) with $\Lambda = (t_2, d_2, r_2, \tilde{a}_\varepsilon, \mu_2, s_2(t_2))$ and $w_0(x) = v(t_2, x)$. Then $s_2(t) \leq g_2(t)$, $v(t,x) \leq w_2(t,x)$ in $D^{s_2}_{t_2}$. So, $g_2(\infty) = \infty$. Similarly to the above,

$$g_2(t) - c_2 t \to \rho_2 \in \mathbb{R}, \quad \|w_2(t,x) - q_2(c_2 t + \rho_2 - x)\|_{L^\infty([0,\, g_2(t)])} \to 0$$

as $t \to \infty$, where $(q_2, c_2)$ is the unique solution of (2) with $\gamma = (\mu_2, r_2\tilde{a}_\varepsilon, r_2, d_2)$. Then $q_2(y) < \tilde{a}_\varepsilon$, and $w_2(t,x) < \tilde{a}_\varepsilon + \varepsilon$ in $D^{g_2}_{t_3}$ for some $t_3 > t_2$. Note $s_2(t) \leq g_2(t)$ for $t \geq t_3$ and $D^{s_2}_{t_3} \subset D^{g_2}_{t_3}$, the following holds:

$$v(t,x) \leq w_2(t,x) < \tilde{a}_\varepsilon + \varepsilon = 1 - h(1-k) + (1+h)\varepsilon \ \text{ in } D^{s_2}_{t_3}.$$

In view of $v(t,x) = 0$ for $x \geq s_2(t)$, and the arbitrariness of $\varepsilon$, it follows that

$$\limsup_{t\to\infty} \max_{[0,\infty)} v(t,\cdot) \leq 1 - h(1-k) := \bar{v}_2.$$

When $h(1-k) \geq 1$, we have $\bar{v}_2 \leq 0$, and so (8) holds since $v(t,x) \geq 0$.

Here we deal with the case $h(1-k) < 1$. There exists $t_4 \gg 1$ such that $v(t,x) \leq \bar{v}_2 + \varepsilon := \bar{v}_2^\varepsilon < 1$ in $[t_4, \infty) \times [0,\infty)$. Obviously, $c_1(\mu_1, r_1(1-k\bar{v}_2^\varepsilon)) > c_2(\mu_2, r_2)$. Similarly to the above, we can get

$$\liminf_{t\to\infty} \min_{[0,s_2(t)]} u(t,\cdot) \geq 1 - k[1-h(1-k)] := \underline{u}_2,$$
$$\limsup_{t\to\infty} \max_{[0,\infty)} v(t,\cdot) \leq 1 - h\underline{u}_2 := \bar{v}_3.$$

If $h\underline{u}_2 \geq 1$, then $\bar{v}_3 = 0$ and (8) holds. If $h\underline{u}_2 < 1$, repeating the above procedure in the way of the proof of [7, Lemma 2.2], we can get (8) eventually.

For any given $0 < \delta \ll 1$, there exists $t_5 \gg 1$ such that $v(t,x) < \delta$ in $[t_5,\infty) \times [0,\infty)$. Obviously, $c_1(\mu_1, r_1(1-k\delta)) > c_2(\mu_2, r_2)$. Replacing $1+\varepsilon$ by $\delta$, similarly to the above we can prove $u(t,x) > 1-\delta$ in $D^{s_2}_{t_6}$ for some $t_6 > t_5$. Therefore, $1 - v - hu < 1 - h(1-\delta) - v < 0$ in $D^{s_2}_{t_6}$ because of $0 < \delta \ll 1$ and $h > 1$. According to [4, Lemma 3.2], $s_2^\infty < \infty$ is followed. This is a contradiction and the proof is finished. $\square$

Theorem 5 is exactly [3, Theorem 3], and our proof is simpler.

Similarly to the proof of [8, Lemma 2.1], it can be shown that

$$0 < s_1(t) \leq K\mu_1 t + s_1^0, \quad \forall\ t > 0,$$

where

$$K = 2\max\left\{\max\{1,\ \|u_0\|_\infty\}\sqrt{r_1/(2d_1)},\ -\min_{[0,\, s_1^0]} u_0'(x)\right\}.$$

**Theorem 6** *Let $d_i, r_i, k, h$ and $\mu_2$ be fixed. Then there exists $0 < \bar{\mu}_1 < \sqrt{2d_2r_2}/K$ such that, when*

$$0 < \mu_1 < \bar{\mu}_1, \quad s_2^0 - s_1^0 > \pi\frac{2d_2}{\sqrt{2d_2r_2 - K^2\mu_1^2}} := L(\mu_1),$$

*we have $s_2(t) \geq K\mu_1 t + s_1^0 + L(\mu_1) \to \infty$ as $t \to \infty$. Moreover, if $k < 1$ and $s_1^0 \geq \tilde{s}_1$, we also have $s_1^\infty = \infty$ for all $\mu_1 > 0$.*

**Proof**. Denote $\sigma = K\mu_1$. For the given $\sigma \in (0, \sqrt{2d_2r_2})$, and these $t$ satisfying $s_2(t) > \sigma t + s_1^0$, we define

$$y = x - \sigma t - s_1^0, \quad w(t,y) = v(t,x), \quad \eta(t) = s_2(t) - \sigma t - s_1^0.$$

Note that $y \geq 0$ implies $x \geq s_1(t)$ and $u(t,x) = 0$ for $x \geq s_1(t)$, we have

$$\begin{cases} w_t - d_2 w_{yy} - \sigma w_y = r_2 w(1-w), & t > 0,\ 0 < y < \eta(t), \\ w(t,0) = v(t, \sigma t + s_1^0), \quad w(t,\eta(t)) = 0, & t \geq 0, \\ w(0,y) = v_0(0, y + s_1^0), & 0 \leq y \leq s_2^0 - s_1^0, \end{cases}$$

and $w(t,y) > 0$ for $t \geq 0$ and $0 < y < \eta(t)$. Let $\lambda$ be the principal eigenvalue of

$$\begin{cases} -d_2\phi'' - \sigma\phi' - r_2\phi = \lambda\phi, \quad 0 < x < \ell, \\ \phi(0) = 0 = \phi(\ell). \end{cases} \tag{11}$$

The following relation between $\lambda$ and $\ell$ holds:

$$\frac{\pi}{\ell} = \frac{\sqrt{4d_2(r_2+\lambda) - \sigma^2}}{2d_2}.$$

Take $\lambda = -r_2/2$ and define

$$\ell_\sigma = \pi\frac{2d_2}{\sqrt{2d_2r_2 - \sigma^2}}, \qquad \phi(y) = \mathrm{e}^{-\frac{\sigma}{2d_2}y}\sin\frac{\pi}{\ell_\sigma}y.$$

Then $(\ell_\sigma, \phi)$ satisfies (11) with $\lambda = -r_2/2$ and $\ell = \ell_\sigma$. Assume $s_2^0 - s_1^0 > \ell_\sigma$. Set

$$\delta_\sigma = \min\left\{\inf_{(0,\,\ell_\sigma)}\frac{w(0,y)}{\phi(y)},\ \frac{1}{2}\inf_{(0,\,\ell_\sigma)}\frac{1}{\phi(y)}\right\}, \quad \psi(y) = \delta_\sigma\phi(y).$$

Then $0 < \delta_\sigma < \infty$. It is easy to see that $\psi(y) \leq w(0,y)$ in $[0,\ell_\sigma]$ and satisfies

$$\begin{cases} -d_2\psi'' - \sigma\psi' \leq r_2\psi(1-\psi), \quad 0 < x < \ell_\sigma, \\ \psi(0) = 0 = \psi(\ell_\sigma). \end{cases}$$

Take a maximal $\bar\sigma \in (0, \sqrt{2d_2r_2})$ so that

$$\sigma < \mu_2\delta_\sigma\frac{\pi}{\ell_\sigma}\exp\left(-\frac{\sigma\ell_\sigma}{2d_2}\right), \quad \forall\, \sigma \in (0,\bar\sigma). \tag{12}$$

For any given $\sigma \in (0,\bar\sigma)$, we claim that $\eta(t) > \ell_\sigma$ for all $t \geq 0$, which implies $s_2(t) \geq \sigma t + s_1^0 + \ell_\sigma \to \infty$. In fact, note $\eta(0) = s_2^0 - s_1^0 > \ell_\sigma$, if our claim is not true, then we can find a $t_0 > 0$ such that $\eta(t) > \ell_\sigma$ for all $0 \leq t < t_0$ and $\eta(t_0) = \ell_\sigma$. Therefore, $\eta'(t_0) \leq 0$, i.e, $s_2'(t_0) \leq \sigma$. On the other hand, by the comparison principle, we have $w(t,y) \geq \psi(y)$ in $[0,t_0] \times [0,\ell_\sigma]$. Particularly, $w(t_0,y) \geq \psi(y)$ in $[0,\ell_\sigma]$. Due to $w(t_0,\ell_\sigma) = 0 = \psi(\ell_\sigma)$, one has

$$w_y(t_0,\eta(t_0)) \leq \psi'(\ell_\sigma) = -\delta_\sigma\frac{\pi}{\ell_\sigma}\exp\left(-\frac{\sigma\ell_\sigma}{2d_2}\right).$$

It follows, upon using $v_x(t_0, s_2(t_0)) = w_y(t_0,\eta(t_0))$, that

$$\sigma \geq s_2'(t_0) = -\mu_2 w_y(t_0,\eta(t_0)) \geq \mu_2\delta_\sigma\frac{\pi}{\ell_\sigma}\exp\left(-\frac{\sigma\ell_\sigma}{2d_2}\right).$$

It is in contradiction with (12).

Take $\bar{\mu}_1 = \bar{\sigma}/K$. Then $0 < \mu_1 < \bar{\mu}_1$ is equivalent to $0 < \sigma < \bar{\sigma}$.

At last, if $k < 1$ and $s_1^0 \geq \tilde{s}_1$, then $s_1^\infty = \infty$ for any $\mu_1 > 0$ by Theorem 4(ii). The proof is complete. $\square$

Theorem 6 can be regarded as an improvement of [3, Theorem 5], here we need neither the assumption $v_0'(x) \leq 0$ in $[s_1^0, s_2^0]$, nor the condition that $d_2$ is suitably large. Moreover, our proof of Theorem 6 is simpler.

From the proof of Theorem 6 it can be seen that if we take $\bar{\sigma} \in (0, \sqrt{d_2 r_2})$ such that (12) holds, then $s_2^\infty = \infty$ is still true provided that $0 < \mu_1 < \bar{\mu}_1$ and $s_2^0 - s_1^0 \geq \pi \frac{2d_2}{\sqrt{d_2 r_2}}$.

Theorem 5 demonstrates that when the superior competitor spreads quickly and the inferior competitor spreads slowly, the inferior competitor will vanish eventually and the superior competitor will spread successfully and occupy the whole space. Take $0 < k < h < 1$ in Theorem 6, the conclusion indicates that if the superior competitor spreads too slow to catch up with the inferior competitor, it may leave enough space for the inferior competitor to survive.

In the following we will discuss the more accurate limits of $(u, v)$ as $t \to \infty$ when $s_1^\infty = s_2^\infty = \infty$. By the comparison principle and [2, Theorem 4.2], it can be deduced that

$$\begin{cases} \liminf\limits_{t\to\infty}(s_1(t)/t) \geq c_1(\mu_1, r_1(1-k)) := \underline{c}_1 & \text{if} \quad k < 1, \\ \liminf\limits_{t\to\infty}(s_2(t)/t) \geq c_2(\mu_2, r_2(1-h)) := \underline{c}_2 & \text{if} \quad h < 1. \end{cases} \tag{13}$$

The following two theorems are the improvements of Theorem 3.

**Theorem 7** *Let $d_i, r_i, \mu_i, k, h$ be fixed and $0 < k, h < 1$. If $s_1^\infty = s_2^\infty = \infty$, then for each $0 < c_0 < \min\{\underline{c}_1, \underline{c}_2\}$,*

$$\lim_{t\to\infty} \max_{[0,\, c_0 t]} |u(t, \cdot) - (1-k)/(1-hk)| = 0, \quad \lim_{t\to\infty} \max_{[0,\, c_0 t]} |v(t, \cdot) - (1-h)/(1-hk)| = 0.$$

**Proof**. According to $0 < c_0 < \min\{\underline{c}_1, \underline{c}_2\}$ and (13), there exist $0 < \sigma_0 \ll 1$ and $t_\sigma \gg 1$ such that

$$c_\sigma := c_0 + \sigma < \min\{\underline{c}_1, \underline{c}_2\}, \quad \forall\ 0 < \sigma \leq \sigma_0; \quad s_1(t),\ s_2(t) > c_\sigma t, \quad \forall\ t \geq t_\sigma.$$

*Step 1*: Similar to the above, the estimate (5) holds for $v$. For any given $0 < \varepsilon \ll 1$, there exists $t_1 > 0$ such that $v(t, x) < 1 + \varepsilon$ in $[t_1, \infty) \times [0, \infty)$. Enlarging $t_1$ if necessary, we may think $s_1(t_1) > \frac{\pi}{2}(\frac{d_1}{r_1 a_\varepsilon})^{1/2}$, where $a_\varepsilon = 1 - k(1+\varepsilon)$.

*Step 2*: Let $(w_1, g_1)$ be the unique solution of (6) with $\Lambda = (t_1, d_1, r_1, a_\varepsilon, \mu_1, s_1(t_1))$ and $w_0(x) = u(t_1, x)$. Then $s_1(t) \geq g_1(t)$, $u(t, x) \geq w_1(t, x)$ in $D_{t_1}^{g_1}$ by the comparison principle. And, $g_1(\infty) = \infty$ since $g_1(t_1) = s_1(t_1) > \frac{\pi}{2}(\frac{d_1}{r_1 a_\varepsilon})^{1/2}$. By use of [10, Theorem 3.1], we get

$$\lim_{t\to\infty}(g_1(t) - c_\varepsilon t) = \rho \in \mathbb{R}, \quad \lim_{t\to\infty} \|w_1(t, x) - q_\varepsilon(c_\varepsilon t + \rho - x)\|_{L^\infty([0, g_1(t)])} = 0,$$

where $(q_\varepsilon, c_\varepsilon)$ is the unique solution of (2) with $\gamma = (\mu_1, r_1 a_\varepsilon, r_1, d_1)$, i.e., $c_\varepsilon = c_1(\mu_1, r_1 a_\varepsilon)$. Note $0 < c_\sigma < \underline{c}_1$, we have $c_\varepsilon > c_\sigma$ as $0 < \varepsilon \ll 1$. Thus, $g_1(t) - c_\sigma t \to \infty$ and $\min_{[0,\, c_\sigma t]}(c_\varepsilon t + \rho - x) \to \infty$ as $t \to \infty$. Similar to the proof of (10) we can derive

$$\liminf_{t\to\infty} \min_{[0,\, c_\sigma t]} u(t, \cdot) \geq 1 - k.$$

There exists $t_2 \gg 1$ such that

$$s_2(t) > c_\sigma t, \quad u(t,x) \geq 1 - k - \varepsilon := b_\varepsilon, \quad \forall\, t \geq t_2,\ 0 \leq x \leq c_\sigma t.$$

*Step 3*: Similar to the arguments of [7, Lemma 2.1], $\limsup_{t\to\infty} v(t,x) \leq 1 - hb_\varepsilon$ uniformly in $x \in [0,1]$. In particular, $v(t,0) \leq 1 - hb_\varepsilon + \varepsilon$ in $[t_3, \infty)$ for some $t_3 > t_2$. Thus, $v$ satisfies

$$\begin{cases} v_t - d_2 v_{xx} \leq r_2 v(1 - hb_\varepsilon - v), & t \geq t_3, \quad 0 < x < c_\sigma t, \\ v(t,0) \leq 1 - hb_\varepsilon + \varepsilon, \quad v < 1 + \varepsilon, & t \geq t_3,\ 0 < x \leq c_\sigma t. \end{cases}$$

We will show that $\limsup_{t\to\infty} \max_{[0,\, c_{\rho\sigma/2} t]} v(t,\cdot) \leq 1 - hb_\varepsilon + \varepsilon$, which leads to

$$\limsup_{t\to\infty} \max_{[0,\, c_{\sigma/2} t]} v(t,\cdot) \leq 1 - h(1-k) := \bar{v}_2 \tag{14}$$

since $\varepsilon > 0$ is arbitrary. To do this, we choose $0 < \delta \ll 1$ and define

$$\varphi(t,x) = 1 - hb_\varepsilon + \varepsilon + hb_\varepsilon \mathrm{e}^{\delta c_\sigma t_3} \mathrm{e}^{\delta(x - c_\sigma t)}, \quad t \geq t_3,\ 0 \leq x \leq c_\sigma t.$$

Evidently,

$$\max_{[0,\, c_{\sigma/2} t]} \varphi(t,\cdot) \leq 1 - hb_\varepsilon + \varepsilon + hb_\varepsilon \mathrm{e}^{\delta c_\sigma t_3} \mathrm{e}^{-\delta \sigma t/2} \to 1 - hb_\varepsilon + \varepsilon$$

as $t \to \infty$, and

$$\varphi(t,0) > 1 - hb_\varepsilon + \varepsilon, \quad \varphi(t, c_\sigma t) \geq 1 + \varepsilon, \quad t \geq t_3; \quad \varphi(t_3, x) \geq 1 + \varepsilon, \quad 0 \leq x \leq c_\sigma t_3.$$

Due to $0 < \varepsilon \ll 1$, we can think of $1 - hb_\varepsilon + \varepsilon > \frac{1}{2}[1 - h(1-k)]$. It is easy to verify that

$$\varphi_t - d_2 \varphi_{xx} \geq r_2 \varphi(1 - hb_\varepsilon - \varphi), \quad t \geq t_3,\ 0 \leq x \leq c_\sigma t$$

provided that $\delta$ satisfies $\delta(c_\sigma + d_2\delta) \leq \frac{r_2}{2}[1 - h(1-k)]$. The comparison principle gives $v(t,x) \leq \varphi(t,x)$ for all $t \geq t_3$ and $0 \leq x \leq c_\sigma t$. So, (14) holds. We write $c_{\sigma/2}$ as $c_\sigma$ for the sake of writing. Then there exists $t_4 > t_3$ such that

$$s_1(t) > c_\sigma t, \quad v(t,x) \leq \bar{v}_2 + \varepsilon := \bar{v}_2^\varepsilon < 1, \quad \forall\, t \geq t_4,\ 0 \leq x \leq c_\sigma t.$$

*Step 4*: Because of $\bar{v}_2^\varepsilon < 1$, we have $c_1(\mu_1, r_1(1 - k\bar{v}_2^\varepsilon)) > c_\sigma$. Take $0 < \mu_1^* < \mu_1$ so that $c_1(\mu_1^*, r_1(1 - k\bar{v}_2^\varepsilon)) = c_\sigma$. Let $(q_\sigma, c_\sigma)$ be the unique solution of (2) with $\gamma = (\mu_1^*, r_1(1 - k\bar{v}_2^\varepsilon), r_1, d_1)$. Owing to $s_1(t) > c_\sigma t$ for all $t \geq t_4$, we can find a function $\tilde{u} \in C^2([0, c_\sigma t_4])$ satisfying $\tilde{u}(x) \leq \min\{q_\sigma(c_\sigma t_4 - x),\, u(t_4, x)\}$ in $[0, c_\sigma t_4]$ and

$$\tilde{u}'(0) = \tilde{u}(c_\sigma t_4) = 0, \ \ \tilde{u}(x) > 0 \ \text{ in } \ [0, c_\sigma t_4).$$

Let $(w_2, g_2)$ be the unique solution of (6) with $\Lambda = (t_4, d_1, r_1, 1 - k\bar{v}_2^\varepsilon, \mu_1^*, c_\sigma t_4)$ and $w_0(x) = \tilde{u}(x)$. Then, by use of [10, Theorem 3.1],

$$g_2(t) - c_\sigma t \to \rho \in \mathbb{R}, \quad \|w_2(t,x) - q_\sigma(c_\sigma t + \rho - x)\|_{L^\infty([0, g_2(t)])} \to 0$$

as $t \to \infty$. Define $z(t,x) = q_\sigma(c_\sigma t - x)$, $\eta(t) = c_\sigma t$. It is easy to verify that

$$\begin{cases} z_t - d_1 z_{xx} = r_1 z(1 - k\bar{v}_2^\varepsilon - z), & t \geq t_4, \quad 0 \leq x \leq \eta(t), \\ -z_x(t,0) > 0, \quad z = 0, \quad \eta'(t) = -\mu_1^* z_x, & t \geq t_4,\ x = \eta(t), \\ \eta(t_4) = g_2(t_4), \quad z(t_4, x) \geq \tilde{u}(x) = w_2(t_4, x), & 0 \leq x \leq c_\sigma t_4. \end{cases}$$

By the comparison principle,

$$g_2(t) \le \eta(t) = c_\sigma t, \quad w_2(t,x) \le z(t,x) = q_\sigma(c_\sigma t - x) \;\text{ in } D^{g_2}_{t_4}.$$

Note that $g_2(t) \le c_\sigma t < s_1(t)$, $w_2(t, g_2(t)) = 0 < u(t, g_2(t))$ in $[t_4, \infty)$, $w_2(t_4, x) \le u(t_4, x)$ in $[0, c_\sigma t_4]$, and

$$u_t - d_1 u_{xx} \ge r_1 u(1 - k\bar{v}_2^\varepsilon - u) \;\text{ in } D^{g_2}_{t_4}.$$

We have $u \ge w_2$ in $D^{g_2}_{t_4}$ by the comparison principle. Similarly to Step 2, it can be derived that

$$\liminf_{t\to\infty} \min_{[0,\, c_{\sigma/2}t]} u(t,\cdot) \ge 1 - k\bar{v}_2 = 1 - k[1 - h(1-k)] := \underline{u}_2.$$

*Step 5*: Define

$$\bar{v}_1 = 1, \quad \underline{u}_1 = 1-k, \quad \bar{v}_n = 1 - h\underline{u}_{n-1}, \quad \underline{u}_n = 1 - k\bar{v}_n, \quad n \ge 2.$$

Then $\underline{u}_n \to \frac{1-k}{1-hk}$, $\bar{v}_n \to \frac{1-h}{1-hk}$ as $n \to \infty$. Repeating the above process we can prove that

$$\liminf_{t\to\infty} \min_{[0,\, c_0 t]} u(t,\cdot) \ge \underline{u}_n, \quad \limsup_{t\to\infty} \max_{[0,\, c_0 t]} v(t,\cdot) \le \bar{v}_n, \quad \forall\; n \ge 1.$$

Consequently,

$$\liminf_{t\to\infty} \min_{[0,\, c_0 t]} u(t,\cdot) \ge (1-k)/(1-hk), \quad \limsup_{t\to\infty} \max_{[0,\, c_0 t]} v(t,\cdot) \le (1-h)/(1-hk).$$

Similarly, we can show that

$$\limsup_{t\to\infty} \max_{[0,\, c_0 t]} u(t,\cdot) \le (1-k)/(1-hk), \quad \liminf_{t\to\infty} \min_{[0,\, c_0 t]} v(t,\cdot) \ge (1-h)/(1-hk).$$

The proof is complete. $\square$

Theorem 7 is an improvement of [9, Theorem 6] in there the condition $hk < 1/2$ is required.

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